

Tidal tensors in the description of gravity and electromagnetism

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November 15, 2011

Abstract

Abstract: In 2008-2009, F. Costa and C. Herdeiro proposed a new gravito - electromagnetic analogy, based on tidal tensors. We show that connections on the tangent bundle of the space-time manifold can help not only in finding a convenient geometrization of their ideas, but also a common mathematical description of the main equations of gravity and electromagnetism.

MSC 2000: 53Z05, 53B05, 53B40, 53C60, 83C22

Keywords: tangent bundle, spray, Ehresmann connection, tidal tensor, Einstein-Maxwell equations

1 Introduction

In two recent papers, [7], [8], F. Costa and C. Herdeiro provided a new gravito-magnetic analogy, meant to overcome the limitations of the two classical ones (namely, the linearized approach, which is only valid in the case of a weak gravitational field and the one based on Weyl tensors, which compares tensors of different ranks). The central role in this analogy is played by worldline deviation equations and the resulting tidal tensors; it is in terms of these tensors that the fundamental equations of the gravitational and electromagnetic fields are expressed and compared. We argue that this is not only a natural idea, but an idea which can be brought further – it can underlie more than just an analogy between the two fields: a common geometric model for these.

Still, in the cited papers, in order to be able to make such an analogy, it is imposed a restriction: in the case of worldline deviation for charged particles in flat Minkowski space, covariant derivatives of the deviation vector field w are required to identically vanish along the initial worldline.

We have shown in a previous paper, [15], that, if we raise to the tangent bundle TM of the space-time manifold and use an appropriate 1-parameter family of Ehresmann connections \tilde{N}^α , we have at least two advantages. On one side, there is no longer need of any restriction upon w – the ”work” of eliminating

the unwanted term in the worldline deviation equation is taken by the adapted frame. On the other side, the obtained tidal tensor expressions for the basic equations of the two physical fields are valid not only in the case when we have either gravity only, or electromagnetic field alone (as in [7], [8]), but also in the general case, when both are present – thus, they also provide a common geometric language for the two physical fields.

The Ehresmann connections we defined in [15] give rise to very convenient frames on TM : adapted frames. As for covariant derivatives of tensors, we will use the ones given by some special affine connections $\overset{\alpha}{D}$ on the tangent bundle. In [15], we defined a first variant of such a 1-parameter family of connections. In the present paper, we propose a different choice for $\overset{\alpha}{D}$, with richer properties: 1) lifts of worldlines of charged particles are autoparallel curves for $\overset{\alpha}{D}$; 2) geodesic deviation equations are as simple as possible; 3) Riemann (and Ricci) tensors of $\overset{\alpha}{D}$ can be obtained from tidal tensors just by differentiating the latter with respect to the fiber coordinates on TM . Also, we discuss in more detail the obtained equations.

The idea we use here – of encoding gravity in the metric and electromagnetism, in Ehresmann connections (together with affine connections) on the tangent bundle¹ TM – was proposed by R. Miron and collaborators, [12], [13], [10]; just, here we use different connections, meant to offer a more convenient expression for worldline deviation equations.

The paper is organized as follows. In Section 2, we present the elements of the gravito-electromagnetic analogy by Costa and Herdeiro ([7], [8]) which are necessary in the subsequent. In the following three sections, we introduce the Ehresmann connections $\overset{\alpha}{N}$, the affine connections $\overset{\alpha}{D}$ and study geodesics, together with geodesic deviation. Section 6 is devoted to the geometric expressions of the basic equations of the two physical fields. In the last section, we rewrite in terms of adapted derivatives Costa and Herdeiro's equations and point out that the restriction imposed in ([7], [8]) upon the deviation vector field is no longer necessary.

2 Tidal tensors and gravito-electromagnetic analogy

Consider a 4-dimensional Lorentzian manifold (M, g) , with signature $(-, +, +, +)$, regarded as space-time manifold, with local coordinates $(x^i)_{i=\overline{0,3}}$ and Levi-Civita connection ∇ . Throughout the paper, we will mean by (∂_i) the natural

¹Other attempts of unifying gravity and electromagnetism, based on tangent bundle geometry, try to include information regarding electromagnetism in Finsler-type metrics (Randers, Beil or Weyl metrics, [5], [6]). Also, recently, Wanas, Youssef and Sid-Ahmed produced another description, [16], based on teleparallelism on TM . Still, we adopted Miron's idea as leading to relatively simple computations and to elegant expressions for Einstein and Maxwell equations.

basis of the module of vector fields on M ; the speed of light c and the gravitational constant k will be considered as equal to 1.

Worldlines of particles subject to *gravity only* are geodesics $s \mapsto (x^i(s))$ of (M, g) :

$$\frac{\nabla u^i}{ds} = 0, \quad u = \dot{x},$$

where s is the natural parameter (i.e., $g_{ij}u^i u^j = -1$). Curvature of space-time becomes manifest in the *geodesic deviation equation*:

$$\frac{\nabla^2 w^i}{ds^2} = e^i_k w^k, \quad e^i_k = r_j^i{}_{kl} u^j u^l, \quad (1)$$

where $w = w^i \partial_i$ is the deviation vector field and e^i_k define the so-called *tidal (electrogravitic) tensor*².

On the other side, in special relativity (where $g_{ij} = \text{diag}(-1, 1, 1, 1)$), the electromagnetic field is described by the 4-*potential* 1-form $A = A_i(x)dx^i$ and the electromagnetic 2-form $F = dA$, i.e.,

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j, \quad F_{ij} = \partial_i A_j - \partial_j A_i. \quad (2)$$

Worldlines of charged particles subject to an electromagnetic field are solutions of the Lorentz equations:

$$\frac{\nabla u^i}{ds} = \frac{q}{m} F^i_j u^j, \quad u = \dot{x}; \quad (3)$$

here, $\frac{\nabla u^i}{ds} = \frac{du^i}{ds}$, q is the electric charge of the particle and m , its mass. For families of worldlines of particles with same ratio $\frac{q}{m}$, one can determine the *worldline deviation equation*:

$$\frac{\nabla^2 w^i}{ds^2} = \frac{q}{m} (E^i_k w^k + F^i_k \frac{\nabla w^k}{ds}), \quad (4)$$

where

$$E^i_k = u^j \nabla_{\partial_k} F^i_j. \quad (5)$$

Traditionally, [7], [2], it is imposed the restriction that, along the initial worldline:

$$\frac{\nabla w^i}{ds} = 0; \quad (6)$$

under this assumption, the worldline deviation equations reduce to:

$$\frac{\nabla^2 w^i}{ds^2} = \frac{q}{m} E^i_k w^k, \quad (7)$$

²Here, we have used a different sign convention for the Riemann tensor ($r_j^i{}_{kl} = \partial_l \gamma_{jk}^i - \partial_k \gamma_{jl}^i + \gamma_{jk}^h \gamma_{hl}^i - \gamma_{jl}^h \gamma_{hk}^i$) than in [7], [8], resulting in a different sign for e^i_j .

which makes it possible to compare (1) and (4). Following the analogy $E^i_k \sim e^i_k$, [7], [8], Maxwell's equations are written (after contracting with the 4-velocity u) as:

$$\begin{aligned} \nabla_{\partial_i} F^{ij} &= 4\pi J^i & \Rightarrow & E^i_i = -4\pi \rho_c \\ \nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} &= 0 & \Rightarrow & E_{[ij]} = \frac{1}{2} u^k \nabla_{\partial_k} F_{ij}, \end{aligned} \quad (8)$$

(where square brackets denote antisymmetrization and $\rho_c = -J_i u^i$), while gravitational field equations take the form:

$$\begin{aligned} r_{ij} &= 8\pi(T_{ij} - \frac{1}{2} g_{ij} T^l_l) & \Rightarrow & e^i_i = -4\pi(2\rho_m - T^i_i) \\ r_{jk} &= r_{kj} & \Rightarrow & e_{[ij]} = 0, \end{aligned} \quad (9)$$

where T_{ij} is the stress-energy tensor and $\rho_m = T_{ij} u^i u^j$.

Remark 1 *Passing to the tangent bundle, we will be able to regard velocities as fiber coordinates and contract with respect to the latter in defining tidal tensors. This way, all the four implications (8), (9) become equivalences (we can "discard" the fiber coordinates from the tidal tensor equations by differentiating with respect to them, thus getting again the classical versions of the equations).*

In the cited papers, the authors also use the analogues of the above tidal tensors, built from the Hodge duals of the 2-forms r and F , resulting in two more pairs of equations (8), (9). Taking into account the above remark, (8) and (9) are sufficient for our purposes, so we have omitted these two extra pairs of equations.

3 Ehresmann connections

Consider now the tangent bundle (TM, π, M) of the 4-dimensional Lorentzian manifold (M, g) ; here, we denote the local coordinates by $(x \circ \pi, y) =: (x^i, y^i)_{i=0,3}$ and by \cdot_i and \cdot^i , partial differentiation with respect to x^i and y^i respectively. An (arbitrary) Ehresmann connection N on TM , [10], [4], gives rise to the adapted basis

$$(\delta_i = \frac{\partial}{\partial x^i} - N^j_i(x, y) \frac{\partial}{\partial y^j}, \quad \dot{\delta}_i = \frac{\partial}{\partial y^i}), \quad (10)$$

and to its dual $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$. For a vector field $X = X^i \delta_i + \tilde{X}^i \dot{\delta}_i$, we denote by $hX = X^i \delta_i$ and $vX = \tilde{X}^i \dot{\delta}_i$ its horizontal and vertical components respectively.

The horizontal 1-form $l = l_i dx^i$, where

$$l^i = \frac{y^i}{\|y\|}, \quad \|y\| = \sqrt{g_{ij} y^i y^j}, \quad (11)$$

(and indices of l are lowered by means of g_{ij}) is called the distinguished section on TM , [4],

A curve $c : t \mapsto (x^i(t))$ on M (where t is an arbitrary parameter) is an autoparallel curve (*geodesic*) for N if:

$$\frac{\delta y^i}{dt} = \frac{dy^i}{dt} + N^i_j(x, y)y^j = 0, \quad y^i = \frac{dx^i}{dt}; \quad (12)$$

By direct computation, it follows that deviations of geodesics (12) are described by:

$$\frac{\delta^2 w^i}{dt^2} = R^i_{jk}(x, y)y^k w^j + \mathbb{T}^i_j(x, y)\frac{\delta w^j}{dt}, \quad (13)$$

where w^i are the components of the deviation vector field (on M),

$$R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k \quad (14)$$

are the components of the curvature of N and

$$\mathbb{T}^i_j = y^k N^i_{k \cdot j} - N^i_j,$$

those of the *strong torsion* of N , [11].

Particular case. A *spray connection*, [1], is an Ehresmann connection with the property that there exist some real-valued functions $G^i = G^i(x, y)$ with the properties:

1. $N^i_j = G^i_{\cdot j} =: G^i_j$;
2. $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\forall \lambda \in \mathbb{R}$;
3. with respect to coordinate changes on TM , the functions G^i behave in such a way that the *spray* $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a vector field.

For spray connections, the strong torsion \mathbb{T} identically vanishes. Conversely, if for a connection N , the strong torsion is identically 0, then, by setting, $2G^i := N^i_k y^k$, it follows that $2G^i_{\cdot j} = N^i_{k \cdot j} y^k + N^i_j = 2N^i_j$, hence, N is a spray connection.

Consequently: spray connections are the only Ehresmann connections with the property that deviations of autoparallel curves are described by equations whose right hand side does not depend on the derivatives of w :

$$\frac{\delta^2 w^i}{dt^2} = E^i_j w^j, \quad (15)$$

where $E^i_j = R^i_{jk} y^k$.

We will call the quantity

$$E = E^i_j \delta_i \otimes dx^j, \quad E^i_j = R^i_{jk} y^k; \quad (16)$$

the *tidal tensor* of N ; for spray connections, this tensor contains all the information regarding deviations of autoparallel curves.

Remark 2 *Geodesic equations for a spray connection can be also written, [1], [10], as:*

$$\frac{dy^i}{dt} + 2G^i(x, y) = 0, \quad y = \dot{x}. \quad (17)$$

Conversely, if equations (17) are globally defined and the functions G^i are 2-homogeneous in y , then, [3], G^i are the coefficients of a spray on TM .

As an application of the above, consider the following 1-parameter family of *Randers-type Finslerian fundamental functions* [4], [13] depending on α :

$$\overset{\alpha}{L} = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} + \alpha A_i \dot{x}^i; \quad (18)$$

here, g_{ij} is the Lorentzian metric on M as above, $A = A_i(x)dx^i$, a 1-form on M (momentarily, with no relation with the one in Section 2) and $\alpha \in \mathbb{R}$ is a parameter.

Extremal curves $x = x(t)$ (i.e., $t = \text{const} \cdot s$) for the action $\int \overset{\alpha}{L} dt$ are given by:

$$\frac{dy^i}{dt} + \gamma^i_{jk} y^j y^k - \alpha \|y\| F^i_j y^j = 0, \quad y^i = \dot{x}^i, \quad (19)$$

where

$$F^i_j = g^{ih}(A_{j,h} - A_{h,j}), \quad \|y\| = \sqrt{g_{ij}y^i y^j}; \quad (20)$$

we get a 1-parameter family of sprays on TM , with coefficients $G^i = \overset{\alpha}{G}^i$ given by:

$$2\overset{\alpha}{G}^i(x, y) = \gamma^i_{jk} y^j y^k + 2\overset{\alpha}{B}^i, \quad (21)$$

where³

$$2\overset{\alpha}{B}^i = -\alpha \|y\| F^i_j y^j. \quad (22)$$

We will also use the notation: $F^i = F^i_j y^j$, i.e., $2\overset{\alpha}{B}^i := -\alpha \|y\| F^i$. If there is no risk of confusion, we will not explicitly indicate in the notation of spray and connection coefficients, adapted derivatives etc., the parameter α (i.e., we will use $G^i, B^i, G^i_j, B^i_j, \dots$ instead of $\overset{\alpha}{G}^i, \overset{\alpha}{B}^i, \overset{\alpha}{G}^i_j, \overset{\alpha}{B}^i_j$ etc.).

It is worth noticing some properties of the functions B^i in (22).

1. Functions B^i , (22), are the components of a horizontal vector field $B = B^i \delta_i$ on TM .

³Indices are raised here by means of the pseudo-Riemannian metric $g_{ij}(x)$. This feature makes our approach different from the classical treatment of Randers geometry (where one uses a more complicated, Finsler-type, metric tensor $g_{ij}(x, y)$, [4]) – and similar to the Ingarden geometry proposed by Miron, [13]; the difference between the Ehresmann connections we build here and those in Ingarden geometry relies in the 2-homogeneity in the fiber coordinates of the term $-\alpha \|y\| F^i_j$ in (19), which allows us to use spray connections.

2. The derivatives of the functions B^i with respect to the fiber coordinates are:

$$\begin{aligned} B^i_{\cdot j} &= B^i_{\cdot j} = -\frac{\alpha}{2}(F^i l_j + \|y\| F^i_{\cdot j}) \\ B^i_{jk} &:= B^i_{\cdot jk} = -\frac{\alpha}{2}(l_{\cdot jk} F^i + l_j F^i_{\cdot k} + l_k F^i_{\cdot j}). \end{aligned} \quad (23)$$

3. From the homogeneity of degree 2 of B in the fiber coordinates, it follows: $B^i_j y^j = 2B^i$, $B^i_{jk} y^k = B^i_{\cdot j}$, $B^i_{\cdot jkl} y^l = 0$.

4. The spray connection coefficients of $N = \overset{\alpha}{N}$ are expressed in terms of γ^i_{jk} and B as:

$$G^i_{\cdot j} = \gamma^i_{jk} y^k + B^i_{\cdot j}. \quad (24)$$

Particular case: For $\alpha = 0$, we get: $\overset{0}{2}G^i = \gamma^i_{jk} y^j y^k$, $\overset{0}{G}^i_{\cdot j} = \gamma^i_{jk} y^k$, hence autoparallel curves of $\overset{0}{N}$ coincide with the geodesics of (M, g) . Moreover, we have

$$\overset{0}{R}^i_{jk} = r^i_{\cdot jk} y^l =: r^i_{jk}$$

and the tidal tensor is in this case, $\overset{0}{E}^i_{\cdot j} = e^i_{\cdot j} = r^i_{jk} y^k$.

4 Affine connections on TM

In the following, we will focus on the functions $\overset{\alpha}{L}$. Consider

$$G^i_{jk} := G^i_{\cdot jk} = \gamma^i_{jk} + B^i_{jk}; \quad (25)$$

we define the affine connections $D = \overset{\alpha}{D}$ on TM which act on the $\overset{\alpha}{N}$ -adapted basis vectors as:

$$D_{\delta_k} \delta_j = G^i_{jk} \delta_i, \quad D_{\delta_k} \dot{\delta}_j = G^i_{jk} \dot{\delta}_i, \quad D_{\dot{\delta}_k} \delta_j = D_{\dot{\delta}_k} \dot{\delta}_j = 0. \quad (26)$$

Remark 3 Connections $\overset{\alpha}{D}$, $\alpha \in \mathbb{R}$, preserve by parallelism the distributions generated by $\overset{\alpha}{N}$ (hence, they are distinguished linear connections, [10], [11], on TM), i.e., for any two vector fields X, Y on TM , we have: $D_X(hY) = hD_X Y$, $D_X(vY) = vD_X Y$.

Connections $\overset{\alpha}{D}$ are generally, non-metrical.

Particular case: For $\alpha = 0$, we get:

$$\overset{0}{G}^i_{jk} = \gamma^i_{jk},$$

i.e., for vector fields X, Y on the base manifold M , the horizontal lift $l_h(\nabla_X Y)$ of the Levi-Civita covariant derivative $\nabla_X Y$ and the $\overset{0}{D}$ -covariant derivative $\overset{0}{D}_{l_h(X)} l_h(Y)$ of the (separately) lifted vector fields coincide:

$$l_h(\nabla_X Y) = \overset{0}{D}_{l_h(X)} l_h(Y).$$

In this sense, $\overset{0}{D}$ can be considered as the TM -equivalent of the Levi-Civita connection ∇ and $\overset{\alpha}{D}$, as a "perturbation" of $\overset{0}{D}$, with contortion tensor B .

Curvature and torsion tensors of $\overset{\alpha}{D}$ can be determined by direct computation.

1. The torsion of $D = \overset{\alpha}{D}$ is given by:

$$\mathbb{T} = R^i_{jk} \dot{\delta}_i \otimes dx^j \otimes dx^k, \quad (27)$$

where R^i_{jk} are the components of the curvature of the Ehresmann connection $\overset{\alpha}{N}$.

2. The curvature of D is:

$$\begin{aligned} \mathbb{R} = & R_j^i{}_{kl} \delta_i \otimes dx^j \otimes dx^k \otimes dx^l + R_j^i{}_{kl} \dot{\delta}_i \otimes \delta y^j \otimes dx^k \otimes dx^l + \\ & + B_j^i{}_{kl} \delta_i \otimes dx^j \otimes dx^k \otimes \delta y^l, \end{aligned} \quad (28)$$

where:

$$R_j^i{}_{kl} = \frac{1}{2}(E^i_k)_{\cdot jl}, \quad B_j^i{}_{kl} = B_{\cdot jkl}. \quad (29)$$

In particular, the Ricci tensor of $\overset{\alpha}{D}$ is given by the Hessian with respect to the fiber coordinates of the trace E^i_i :

$$R_{jl} = -\frac{1}{2}(E^i_i)_{\cdot jl} = R_{jli}. \quad (30)$$

3. Conversely, the tidal tensor E of $\overset{\alpha}{N}$ and its trace are obtained⁴ in terms of \mathbb{R} as:

$$E^i_k = R_j^i{}_{kl} y^j y^l, \quad E^i_i = -R_{jl} y^j y^l. \quad (31)$$

In particular, for $\alpha = 0$, we have:

$$\overset{0}{E}^i_k = r_j^i{}_{kl} y^j y^l = e^i_k, \quad \overset{0}{E}^i_i = e^i_i.$$

⁴Expression (31) points out an almost complete similarity between the tidal tensor and the notion of flag curvature in Finsler geometry. The difference consists in the metric tensor used in raising and lowering indices, which is not the Finslerian one corresponding to $\overset{\alpha}{L}$ – and which leads to somehow different properties.

5 Geodesics and geodesic deviations

Consider a 1-parameter variation $c = c(t, \varepsilon)$, $c(t, 0) = x(t)$, of a curve $t \mapsto (x^i(t))$ on M . We denote by V the complete lift of the velocity $\dot{x}^i(t)\partial_i$ to TM , expressed in the adapted basis as:

$$V := y^i \delta_i + \frac{\delta y^i}{dt} \dot{\delta}_i, \quad y^i = \frac{dx^i}{dt} \quad (32)$$

and by $\frac{D}{dt}$, the covariant derivative D_V . The complete lift of the deviation vector field w along the curve $t \mapsto (x^i(t))$ is:

$$W = w^i \delta_i + \frac{\delta w^i}{dt} \dot{\delta}_i.$$

In the previous section, we have actually proved that, for extremal curves of $\overset{\alpha}{L}$, the complete lift V is horizontal (with respect to $\overset{\alpha}{N}$): $V = hV$. These curves can be also characterized in terms of covariant derivatives attached to $\overset{\alpha}{D}$.

Proposition 4 1. *Extremal curves for the Randers-type Lagrangian $\overset{\alpha}{L}$ obey the equation:*

$$D_V(hV) = 0. \quad (33)$$

2. *Deviations of geodesics (33) are given by:*

$$D_V^2(hW) = R(V, W)(hV); \quad (34)$$

in local coordinates, this is:

$$\frac{D^2 w^i}{dt^2} = E^i_k w^k, \quad (35)$$

here, all covariant derivatives are considered "with reference vector y ", [4], i.e., in their local expressions, $G^i_j = G^i_j(x, y)$, $G^i_{jk} = G^i_{jk}(x, y)$.

Proof. 1) Along the curve $t \mapsto (x^i(t))$, we have:

$$D_V(hV) = \frac{D}{dt}(y^i \delta_i) = \left(\frac{dy^i}{dt} + G^i_{jk} y^j y^k\right) \delta_i = \left(\frac{dy^i}{dt} + G^i_j y^j\right) \delta_i = \frac{\delta y^i}{dt} \delta_i,$$

where $y^i = \dot{x}^i$. The extremality condition $\frac{\delta y^i}{dt} = 0$ is therefore equivalent to $D_V(hV) = 0$.

2) Differentiating (33) with respect to W , we get:

$$0 = D_W D_V(hV) = D_V D_W(hV) + R(W, V)(hV) \quad (36)$$

(where we have taken into account that $[V, W] \equiv [\frac{\partial}{\partial t}, \frac{\partial}{\partial \varepsilon}] = 0$); further, $D_W(hV) = hD_WV = hD_VW + hT(W, V) = D_V(hW)$ (since $T(W, V)$ is vertical). Thus, (36) becomes:

$$D_V^2(hW) + R(W, V)(hV) = 0,$$

which immediately yields (34). Expressing the term $R(V, W)(hV)$ in local coordinates and taking into account that $B^i_{jkl}y^l = 0$, we get (35). ■

6 Basic equations of gravitational and electromagnetic fields

In the following, we will apply the above construction to the case when g_{ij} describes the gravitational field and $A = A_i dx^i$ in (18), is the 4-potential of the electromagnetic field. The differential forms A and F , 2), can be lifted to horizontal forms on TM , which we will denote in the same manner. Unless elsewhere specified, the parameter $\alpha \neq 0$ is arbitrary.

Consider the *angular metric*, [4]: $h = h_{ij} dx^i \otimes dx^j$ (regarded as a horizontal tensor on TM) given by:

$$h_{ij} = g_{ij} - l_i l_j.$$

The angular metric has the properties:

$$h_{ij} = \|y\| l_{i,j}, \quad h_{ij} y^j = 0. \quad (37)$$

The electromagnetic 2-form F can be expressed in terms of the δ_j -covariant derivatives of the 1-form $l = l_i dx^i$. More precisely, we have:

$$D_{\delta_j} l_i = \overset{0}{D}_{\delta_j} l_i - B^k_{j \cdot k} l_i - B^k_{i j} l_k. \quad (38)$$

It is easy to see that $\overset{0}{D}_{\delta_j} l_i = 0$. Evaluating the remaining terms with the help of (23) and taking into account (37), we get that:

$$D_{\delta_j} l_i = \frac{\alpha}{2} F_{ij}; \quad (39)$$

the above can be also written in the form (which reminds [13], [10], [12])

$$\alpha F_{ij} = D_{\delta_j} l_i - D_{\delta_i} l_j, \quad (40)$$

in coordinate-free notation: $\alpha F = h(dl)$.

We are now able to express Einstein-Maxwell equations in terms of tidal tensors attached to $D = \overset{\alpha}{D}$, $\alpha \neq 0$.

A. Homogeneous Maxwell equations

Using (40) and Ricci identities, [10], in order to commute double covariant derivatives of l , we obtain:

$$-\alpha \|y\| (D_{\delta_k} F_{ij} + D_{\delta_j} F_{ki} + D_{\delta_i} F_{jk}) y^k = \tilde{E}_{[ij]}, \quad (41)$$

where:

$$\tilde{E}_{ij} = h_{ik} E_j^k. \quad (42)$$

Expressing $D = \overset{0}{D} + B$, we notice that the terms involving the contortion B cancel each other, hence the above relation is actually:

$$\tilde{E}_{[ij]} = -\alpha \|y\| (\overset{0}{D}_{\delta_k} F_{ij} + \overset{0}{D}_{\delta_j} F_{ki} + \overset{0}{D}_{\delta_i} F_{jk}) y^k. \quad (43)$$

Since $F_{ij} = F_{ij}(x)$ is projectable to M , we can write (43) as:

$$\tilde{E}_{[ij]} = -\alpha \|y\| (\nabla_{\delta_k} F_{ij} + \nabla_{\delta_j} F_{ki} + \nabla_{\delta_i} F_{jk}) y^k.$$

We have thus proven:

Proposition 5 *Homogeneous Maxwell equations are written in terms of the tidal tensor E as:*

$$\tilde{E}_{[ij]} = 0, \quad (44)$$

with \tilde{E} as in (42).

B. Inhomogeneous Maxwell equations

Decomposing the curvature R^i_{jk} in terms of $r^i_{jk} = \overset{0}{R}^i_{jk}$ and B^i_j and contracting by y^k , we get by direct computation that:

$$E^i_i = e^i_i - \overset{0}{D}_{\delta_i} (2B^i) + B^l_i B^i_l. \quad (45)$$

The derivative term is: $-\overset{0}{D}_{\delta_i} (2B^i) = \overset{0}{D}_{\delta_i} (\alpha \|y\| F^i_j y^j) = \alpha \|y\| y^j \overset{0}{D}_{\delta_i} F^i_j = \alpha \|y\| y^j \nabla_{\delta_i} F^i_j$.

As a consequence, we get:

Proposition 6 *Inhomogeneous Maxwell equations are expressed in terms of tidal tensors as:*

$$E^i_i = e^i_i - 4\pi\alpha\rho_c \|y\|^2 + B^l_i B^i_l, \quad (46)$$

where $\rho_c = -J^i l_i$ and $e^i_i = \overset{0}{E}^i_i$.

Remark 7 1. An alternative expression can be obtained if we notice that

$$B^i = \frac{1}{2} g^{ih} \delta_h (\|y\|^2) =: \frac{1}{2} \delta^i \|y\|^2; \text{ we get: } \overset{0}{D}_{\delta_i} B^i - B^h_i B^i_h = D_{\delta_i} B^i = \frac{1}{2} D_{\delta_i} (\delta^i \|y\|^2), \text{ i.e., we have the equality:}$$

$$E^i_i = e^i_i - 2\pi\alpha\rho_c \|y\|^2 - \frac{1}{2} D_{\delta_i} (\delta^i \|y\|^2). \quad (47)$$

2. The trace of \tilde{E} is: $\tilde{E}^i_i = g^{ij}\tilde{E}_{ji} = g^{ij}h^k_j E_{ki} = g^{ij}(\delta^k_j - l^k l_j)E_{ki} = E^i_i - l^k l^i E_{ki}$.

Taking into account relation (31), we obtain that $l^k l^i E_{ki} = 0$, which means that $\tilde{E}^i_i = E^i_i$. In other words, we can use in (46) either of the versions \tilde{E}^i_i or E^i_i .

C. Einstein field equations

Einstein field equations can also be obtained from the trace E^i_i , (45), if we substitute, this time, the term $e^i_i = \overset{0}{E}^i_i$. Contracted with $y^i y^j$, the Einstein field equations $r_{ij} = 8\pi(T_{ij} - \frac{1}{2}T^l_l g_{ij})$ become:

$$e^i_i = -8\pi(T_{ij}y^i y^j - \frac{1}{2}T^l_l \|y\|^2). \quad (48)$$

The stress energy tensor T_{ij} can be decomposed as: $T_{ij} = \overset{em}{T}_{ij} + \overset{m}{T}$, where $\overset{em}{T}_{ij}$ is the stress-energy tensor of the electromagnetic field, [9]:

$$\overset{em}{T}_{ij} = \frac{1}{4\pi}(-F^h_i F_{hj} + \frac{1}{4}F^{kh} F_{kh} g_{ij})$$

and $\overset{m}{T}_{ij}$ is the stress-energy tensor of matter (and/or other fields). The electromagnetic stress-energy tensor $\overset{em}{T}_{ij}$ has zero trace $\overset{em}{T}^l_l = 0$, hence, in (48), $T^l_l = \overset{m}{T}^l_l$.

On the other side, taking in (39) covariant derivative by δ_k and successively contracting by g^{jk} and $\|y\|^2 l^i$, we obtain the identity:

$$\|y\|^2 l^i \square l_i = \overset{0}{D}_{\delta_i} B^i + \alpha^2(-F^h F_h + \frac{1}{4}F^{kh} F_{kh} \|y\|^2),$$

where $\square l_i := g^{jk} D_{\delta_k} D_{\delta_j} l_i$. The latter equation is actually: $4\pi\alpha^2 \overset{em}{T}_{ij} y^i y^j = \|y\|^2 l^i \square l_i - \overset{0}{D}_{\delta_i} B^i$.

Consequently, relation (48) is equivalent to:

$$e^i_i = -\frac{2}{\alpha^2}(\|y\|^2 l^i \square l_i - \overset{0}{D}_{\delta_i} B^i) - 8\pi(\overset{m}{T}_{ij} y^i y^j - \frac{1}{2}\overset{m}{T}^l_l \|y\|^2), \quad (49)$$

Replacing into the expression (45) of E^i_i and denoting $\rho_m := \overset{m}{T}_{ij} l^i l^j$, we finally have:

$$\frac{1}{\|y\|^2} E^i_i = \frac{2}{\|y\|^2} \{(\frac{1}{\alpha^2} - 1) \overset{0}{D}_{\delta_i} B^i + B^i_l B^l_i\} - \frac{2}{\alpha^2} l^i \square l_i - 8\pi(\rho_m - \frac{1}{2}\overset{m}{T}^l_l). \quad (50)$$

D. Equations of motion of charged particles

Equations of motion of a charged particle,[9], are nothing but (19):

$$\frac{\overset{\alpha}{D}y^i}{dt} = 0, \quad y = \dot{x}, \quad (51)$$

in which, this time, we set:

$$\alpha = \frac{q}{m}. \quad (52)$$

For particles having the same ratio $\frac{q}{m}$, worldline deviation equations are given by

$$\frac{\overset{\alpha}{D}^2 w^i}{dt^2} = E^i_j w^j, \quad \alpha = \frac{q}{m}. \quad (53)$$

7 Particular cases

A. Gravity only:

In this case, we have $B^i = 0$, which means that all the affine connections $\overset{\alpha}{D}$, $\alpha \in \mathbb{R}$, have vanishing contortion (they actually coincide) and $R_j^i{}_{kl} = r_j^i{}_{kl}$. The tidal tensor is given by $e^i_j = r_l^i{}_{jk} y^l y^k$ and equations (9) become:

$$\begin{aligned} r_{ij} &= 8\pi \left(T_{ij}^m - \frac{1}{2} g_{ij} T_l^l \right) & \Leftrightarrow & \quad \frac{1}{\|y\|^2} e^i_i = -4\pi(2\rho_m - T^i_i) \\ r_{jk} &= r_{kj} & \Leftrightarrow & \quad \tilde{e}_{[ij]} = 0, \end{aligned} \quad (54)$$

where, this time, $\rho_m = T_{ij} l^i l^j$.

B. Electromagnetism in flat Minkowski space

In this case, we have $\gamma^i_{jk} = 0$, $e^i_j = 0$ and $G^i_{jk} = B^i_{jk}$. In the expression (28) of the curvature of $\overset{\alpha}{D}$ ($\alpha \neq 0$), the components $R_j^i{}_{kl}$, $B_j^i{}_{kl}$ only depend on B .

Maxwell equations are written in terms of tidal tensors as:

$$\begin{aligned} \nabla_{\partial_i} F^{ij} &= 4\pi J^i & \Leftrightarrow & \quad \frac{1}{\|y\|^2} E^i_i = -4\pi\alpha\rho_c + \frac{1}{\|y\|^2} B^i_h B^h_i \\ \nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} &= 0 & \Leftrightarrow & \quad \tilde{E}_{[ij]} = 0. \end{aligned} \quad (55)$$

Thus, we found analogous equations to the ones determined by Costa and Herdeiro, [7], [8], without resorting to any restriction upon the derivatives of the deviation vector w .

Acknowledgment. The work was supported by the Sectorial Operational Program Human Resources Development (SOP HRD), financed from the European Social Fund and by Romanian Government under the Project number POSDRU/89/1.5/S/59323.

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